

Hölder Continuity of the Solution for a Class of Nonlinear SPDE Arising from One Dimensional Superprocesses

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Abstract

The Hölder continuity of the solution $X_t(x)$ to a nonlinear stochastic partial differential equation (see (1.2) below) arising from one dimensional super process is obtained. It is proved that the Hölder exponent in time variable is as close as to $1/4$, improving the result of $1/10$ in [3]. The method is to use the Malliavin calculus. The Hölder continuity in spatial variable x of exponent $1/2$ is also obtained by using this new approach. This Hölder continuity result is sharp since the corresponding linear heat equation has the same Hölder continuity.

1 Introduction

Consider a system of particles indexed by multi-indexes α in a random environment whose motions are described by

$$x_\alpha(t) = x_\alpha + B^\alpha(t) + \int_0^t \int_{\mathbb{R}} h(y - x_\alpha(u)) W(du, dy), \quad (1.1)$$

where $h \in L^2(\mathbb{R})$, $(B^\alpha(t); t \geq 0)_\alpha$ are independent Brownian motions and W is a Brownian sheet on $\mathbb{R}_+ \times \mathbb{R}$ independent of B^α . For more detail about this model, we refer to Wang ([8], [9]) and Dawson, Li and Wang [1]. Under some specifications for the branching mechanism and in the limiting situation, Dawson, Vaillancourt and Wang [2] obtained that the density of the branching particles satisfies the following stochastic partial differential equation (SPDE):

$$\begin{aligned} X_t(x) = & \mu(x) + \int_0^t \Delta X_u(x) dr - \int_0^t \int_{\mathbb{R}} \nabla_x (h(y - x) X_u(x)) W(du, dy) \\ & + \int_0^t \sqrt{X_u(x)} \frac{V(du, dx)}{dx}, \end{aligned} \quad (1.2)$$

*David Nualart is supported by the NSF grant DMS0604207.

Keywords: nonlinear stochastic partial differential equation, stochastic heat kernel, conditional probability density in a random environment, Malliavin calculus, Hölder continuity, moment estimates.

where V is a Brownian sheet on $\mathbb{R}_+ \times \mathbb{R}$ independent of W . The joint Hölder continuity of $(t, x) \mapsto X_t(x)$ is left as an open problem in [2].

Let $H_2^k(\mathbb{R}) = \{u \in L^2(\mathbb{R}); u^{(i)} \in L^2(\mathbb{R}) \text{ for } i = 1, 2, \dots, k\}$, the Sobolev space with norm $\|h\|_{k,2}^2 = \sum_{i=0}^k \|h^{(i)}\|_{L^2(\mathbb{R})}^2$. In a recent paper, Li, Wang, Xiong and Zhou [3] proved that $X_t(x)$ is almost surely jointly Hölder continuous, under the condition that $h \in H_2^2(\mathbb{R})$ with $\|h\|_{1,2}^2 < 2$ and $X_0 = \mu \in H_2^1(\mathbb{R})$ is bounded. More precisely, they showed that for fixed t its Hölder exponent in x is in $(0, 1/2)$ and for fixed x its Hölder exponent in t is in $(0, 1/10)$. Comparing to the Hölder continuity for the stochastic heat equation which has the Hölder continuity of $1/4$ in time, it is conjectured that the Hölder continuity of $X_t(x)$ should also be $1/4$.

The aim of this paper is to provide an affirmative answer to the above conjecture. Here is the main result of this paper.

Theorem 1.1 *Suppose that $h \in H_2^2(\mathbb{R})$ and $X_0 = \mu \in L^2(\mathbb{R})$ is bounded. Then the solution to $X_t(x)$ is jointly Hölder continuous with the Hölder exponent in x in $(0, 1/2)$ and with the Hölder exponent in t in $(0, 1/4)$. That is, for any $0 \leq s < t \leq T$, $x, y \in \mathbb{R}$ and $p \geq 1$, there exists a constant C depending only on p, T , $\|h\|_{2,2}$ and $\|\mu\|_{L^2(\mathbb{R})}$ such that*

$$E |X_t(y) - X_s(x)|^{2p} \leq C(1 + t^{-p})(|x - y|^{p-\frac{1}{2}} + (t - s)^{\frac{p}{2}-\frac{1}{4}}). \quad (1.3)$$

Note that the term t^{-p} in the right hand side of (1.3) implies that the Hölder norm of $X_t(x)$ blows up as $t \rightarrow 0$. This problem arises naturally since we only assume $X_0 = \mu \in L^2(\mathbb{R})$.

When $h = 0$ the equation (1.2) is reduced to the famous Dawson-Watanabe equation (process). The study on the joint Hölder continuity for this equation has been studied by Konno and Shiga [4] and Reimers [7]. The starting point is to interpret the equation (when $h = 0$) in mild form with the heat kernel associated with the Laplacian Δ in (1.2). Then the properties of the heat kernel (Gaussian density) can be fully used to analyze the Hölder continuity.

The straightforward extension of the mild solution concept and technique to general nonzero h case in (1.2) meets a substantial difficulty. To overcome this difficulty, Li et al [3] replace the heat kernel by a random heat kernel associated with

$$\int_0^t \Delta X_u(x) dr - \int_0^t \int_{\mathbb{R}} \nabla_x(h(y-x) X_u(x)) W(du, dy).$$

The random heat kernel is given by the conditional transition function of a typical particle in the system with W given. To be more precise, consider the spatial motion of a typical particle in the system:

$$\xi_t = \xi_0 + B_t + \int_0^t \int_{\mathbb{R}} h(y - \xi_u) W(du, dy), \quad (1.4)$$

where $(B_t; t \geq 0)$ is a Brownian motion. For $r \leq t$ and $x \in \mathbb{R}$, define the conditional (conditioned by W) transition probability by

$$P_t^{r,x,W}(\cdot) \equiv P^W(\xi_t \in \cdot | \xi_r = x). \quad (1.5)$$

Denote by $p^W(r, x; t, y)$ the density of $P_t^{r, x, W}(\cdot)$. It is proved that $X_t(y)$ has the following convolution representation:

$$\begin{aligned} X_t(y) &= \int_{\mathbb{R}} \mu(z) p^W(0, z; t, y) dz + \int_0^t \int_{\mathbb{R}} p^W(r, z; t, y) Z(dr, dz) \\ &\equiv X_{t,1}(y) + X_{t,2}(y), \end{aligned} \quad (1.6)$$

where $Z(dr, dz) = \sqrt{X_r(z)} V(dr, dz)$. Then they introduce a fractional integration by parts technique to obtain the Hölder continuity estimates, using Krylov's L_p theory (cf. Krylov [5]) for linear SPDE.

In this paper, we shall use the techniques from Malliavin calculus to obtain more precise estimates for the conditional transition function $p^W(r, x; t, y)$. This allows us to improve the Hölder continuity in the time variable for the solution $X_t(x)$.

The rest of the paper is organized as follows: In Section 2, we briefly recall some notations and results on Malliavin calculus. Then we derive moment estimates for the conditional transition function in Section 3. We study the Hölder continuity in spatial and time variables of $X_t(x)$ in Section 4 and Section 5 respectively. The proof of Theorem 1.1 is concluded in Section 5.

Along the paper, we shall use the following notations; $\|\cdot\|_H$ denotes the norm on Hilbert space $H = L^2([0, T])$, $\|\cdot\|$ (and $\|\cdot\|_p$) denotes the norm on $L^2(\mathbb{R})$ (and on $L^p(\Omega)$). The expectation on (Ω, \mathcal{F}, P) is denoted by E and the conditional expectation with respect to the process W is denoted by E^B .

We denote by C a generic positive constant depending only on $p, T, \|h\|_{2,2}$ and $\|\mu\|_{L^2(\mathbb{R})}$.

2 Preliminaries

Fix a time interval $[0, T]$. Let $(B_t; t \geq 0)$ be a standard Brownian motion. Let \mathcal{S} denote the class of smooth random variables of the form $F = f(B_{t_1}, \dots, B_{t_n})$, where $t_1, \dots, t_n \in [0, T]$, $n \geq 1$, and $f \in C_p^\infty(\mathbb{R}^n)$, the set of smooth functions f such that f itself and all its partial derivatives have at most polynomial growth. Given $F = f(B_{t_1}, \dots, B_{t_n})$ in \mathcal{S} , its Malliavin derivative DF is the H -valued ($H = L^2([0, T])$) random variable given by

$$D_t F = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(B_{t_1}, \dots, B_{t_n}) \mathbf{1}_{[0, t_i]}(t).$$

The derivative operator D is a closable and unbounded operator on $L^2(\Omega)$ taking values in $L^2(\Omega, H)$. For any $p \geq 1$, we denote by $\mathbb{D}^{1,p}$ the closure of \mathcal{S} with respect to the norm $\|\cdot\|_{1,p}$ given by:

$$\|DF\|_{1,p}^p = E(|F|^p) + E(\|DF\|_H^p).$$

We denote by δ the adjoint operator of D , which is an unbounded from a domain of $L^2(\Omega, H)$ to $L^2(\Omega)$. In particular, if $u \in \text{Dom}(\delta)$, then $\delta(u)$ is characterized by the following duality relation:

$$E(\delta(u)F) = E(\langle DF, u \rangle_H) \quad \text{for any } F \in \mathbb{D}^{1,2}.$$

The operator δ is called the *divergence* operator. The following two lemmas are from [6], Propositions 1.5.4 and 2.1.1 and are used frequently in this paper.

Lemma 2.1 *The divergence operator δ is continuous from $\mathbb{D}^{1,p}(H)$ to $L^p(\Omega)$, for any $p > 1$. That is, there exists a constant C_p such that*

$$\|\delta(u)\|_{L^p(\Omega)} \leq C_p \left(\|Eu\|_H + \|Du\|_{L^p(\Omega, H \otimes H)} \right). \quad (2.1)$$

Lemma 2.2 *Let F be a random variable in the space $\mathbb{D}^{1,2}$, and suppose that $\frac{DF}{\|DF\|_H^2}$ belongs to the domain of the operator δ in $L^2(\Omega)$. Then the law of F has a continuous and bounded density given by*

$$p(x) = E \left[\mathbf{1}_{\{F > x\}} \delta \left(\frac{DF}{\|DF\|_H^2} \right) \right].$$

From $E\delta(u) = 0$ for any $u \in \text{Dom}(\delta)$ and the Hölder inequality it follows that

Lemma 2.3 *Let F be a random variable and let $u \in \mathbb{D}^{1,q}(H)$ with $q > 1$. Then for the conjugate pair p and q (i.e. $\frac{1}{p} + \frac{1}{q} = 1$),*

$$|E[\mathbf{1}_{\{F > x\}} \delta(u)]| \leq (P(|F| > |x|))^{\frac{1}{p}} \|\delta(u)\|_{L^q(\Omega)}. \quad (2.2)$$

3 Moment estimates

In this section, we derive moment estimates for the derivatives of ξ_t and the conditional transition function $p^W(r, x; t, y)$.

Recall that $\xi_t = \xi_t^{r,x}$ with initial value $\xi_r = x$ is given by

$$\xi_t = x + B_r^t + I_r^t(h), \quad 0 \leq r < t \leq T, \quad (3.1)$$

where we introduced the notations

$$B_r^t \equiv B_t - B_r, \text{ and } I_r^t(h) \equiv \int_r^t \int_{\mathbb{R}} h(y - \xi_u) W(du, dy). \quad (3.2)$$

Since $h \in H_2^2(\mathbb{R})$, by using the standard Picard iteration scheme, we can prove that such a solution ξ_t to the stochastic differential equation (3.1) exists, and by a regularization argument of h we can prove that $\xi_t \in \mathbb{D}^{2,2}$ (here the Malliavin derivative is with respect to B). Taking the Malliavin derivative D_θ with respect to B , we have

$$D_\theta \xi_t = \mathbf{1}_{[r,t]}(\theta) \left[1 - \int_\theta^t \int_{\mathbb{R}} h'(y - \xi_u) D_\theta \xi_u W(du, dy) \right]. \quad (3.3)$$

Note that

$$M_{\theta,t} := \int_\theta^t \int_{\mathbb{R}} h'(y - \xi_u) W(du, dy)$$

is a martingale with quadratic variation $\langle M \rangle_{\theta,t} = \|h'\|^2(t - \theta)$ for $t > \theta$. Thus

$$D_\theta \xi_t = \mathbf{1}_{[r,t]}(\theta) \exp \left(M_{\theta,t} - \frac{1}{2} \|h'\|^2(t - \theta) \right). \quad (3.4)$$

As a result, we have

$$D_\eta D_\theta \xi_t = \mathbf{1}_{[r,t]}(\theta) \exp\left(M_{\theta,t} - \frac{1}{2} \|h'\|^2(t-\theta)\right) D_\eta M_{\theta,t} = D_\theta \xi_t \cdot D_\eta M_{\theta,t}, \quad (3.5)$$

where $D_\eta M_{\theta,t} = \mathbf{1}_{[\theta,t]}(\eta) \int_\eta^t \int_{\mathbb{R}} h''(y - \xi_u) D_\eta \xi_u W(du, dy)$.

The next lemma gives estimates for the moments of $D\xi_t$ and $D^2\xi_t$.

Lemma 3.1 *For any $0 \leq r < t \leq T$ and $p \geq 1$, we have*

$$\| \|D\xi_t\|_H \|_{2p} \leq \exp\left((2p-1) \|h'\|^2(t-r)\right) (t-r)^{\frac{1}{2}}, \quad (3.6)$$

$$\| \|D^2\xi_t\|_{H \otimes H} \|_{2p} \leq C_p \|h''\| \exp\left((4p-1) \|h'\|^2(t-r)\right) (t-r)^{\frac{3}{2}}, \quad (3.7)$$

and for any $\gamma > 0$,

$$E(\|D\xi_t\|_H^{-2\gamma}) \leq \exp\left((2\gamma^2 + \gamma) \|h'\|^2(t-r)\right) (t-r)^{-\gamma}. \quad (3.8)$$

Proof. Note that for any $p \geq 1$ and $r \leq \theta < t$,

$$\begin{aligned} \|D_\theta \xi_t\|_{2p}^2 &= \left(E \exp \left[2p \left(M_{\theta,t} - \frac{1}{2} \|h'\|^2(t-\theta) \right) \right] \right)^{\frac{1}{p}} \\ &= \exp \left((2p-1) \|h'\|^2(t-\theta) \right). \end{aligned} \quad (3.9)$$

Then (3.6) follows from Minkowski's inequality and (3.9) since

$$\| \|D\xi_t\|_H \|_{2p}^2 = \left[E \left(\int_r^t |D_\theta \xi_t|^2 d\theta \right)^p \right]^{\frac{1}{p}} \leq \int_r^t \|D_\theta \xi_t\|_{2p}^2 d\theta.$$

Applying the Burkholder-Davis-Gundy inequality we have for any $r \leq \theta \leq \eta < t$

$$\begin{aligned} \|D_\eta M_{\theta,t}\|_{2p}^2 &\leq C_p \left(E \left| \int_\eta^t \int_{\mathbb{R}} |h''(y - \xi_u) D_\eta \xi_u|^2 dy du \right|^p \right)^{\frac{1}{p}} \\ &\leq C_p \|h''\|^2 \int_\eta^t \|D_\eta \xi_u\|_{2p}^2 du. \end{aligned} \quad (3.10)$$

Combining (3.5), (3.9) and (3.10) yields for any $r \leq \theta \leq \eta < t$

$$\begin{aligned} \|D_\eta D_\theta \xi_t\|_{2p}^2 &= \|D_\theta \xi_t D_\eta M_{\theta,t}\|_{2p}^2 \leq \|D_\theta \xi_t\|_{4p}^2 \|D_\eta M_{\theta,t}\|_{4p}^2 \\ &\leq C_p \|h''\|^2 \exp\left(2(4p-1) \|h'\|^2(t-\theta)\right) (t-\eta). \end{aligned} \quad (3.11)$$

An application of Minkowski's inequality implies that

$$\| \|D^2\xi_t\|_{H \otimes H} \|_{2p}^2 \leq \int_r^t \int_r^t \|D_\eta D_\theta \xi_t\|_{2p}^2 d\theta d\eta.$$

This yields (3.7).

For the negative moments of $\|D\xi_t\|_H$, by Jensen's inequality we have

$$E(\|D\xi_t\|_H^{-2\gamma}) = E\left(\int_r^t |D_\theta \xi_t|^2 d\theta\right)^{-\gamma} \leq (t-r)^{-\gamma-1} \int_r^t E|D_\theta \xi_t|^{-2\gamma} d\theta.$$

Then, (3.8) follows immediately. ■

The moment estimates of the Malliavin derivatives of the difference $\xi_t - \xi_s$ can also be obtained in a similar way. The next lemma gives these estimates.

Lemma 3.2 *For $0 \leq s < t \leq T$ and $p \geq 1$, we have*

$$\| \|D(\xi_t - \xi_s)\|_H \|_{2p} < C(t-s)^{\frac{1}{2}}, \quad (3.12)$$

and

$$\| \|D^2(\xi_t - \xi_s)\|_{H \otimes H} \|_{2p} < C(t-s)^{\frac{3}{2}}. \quad (3.13)$$

Proof. Similar to (3.3), we have

$$\begin{aligned} D_\theta \xi_t &= D_\theta \xi_s + \mathbf{1}_{[s,t]}(\theta) - \int_{\theta \vee s}^t \int_{\mathbb{R}} h'(y - \xi_u) D_\theta \xi_u W(du, dy) \\ &= D_\theta \xi_s + \mathbf{1}_{[s,t]}(\theta) - I_\theta^t(h' D_\theta \xi.), \end{aligned} \quad (3.14)$$

where henceforth for any process $Y = (Y_t, 0 \leq t \leq T)$ and $f \in L^2(\mathbb{R})$, we denote

$$I_\theta^t(fY) = \mathbf{1}_{[s,t]}(\theta) \int_\theta^t \int_{\mathbb{R}} f(y - \xi_u) Y_u W(du, dy).$$

Applying the Burkholder-Davis-Gundy inequality with (3.9), we obtain for $s \leq \theta \leq t$

$$\begin{aligned} \|I_\theta^t(h' D_\theta \xi.)\|_{2p}^2 &\leq \left(E \left| \int_\theta^t \int_{\mathbb{R}} |h'(y - \xi_u) D_\theta \xi_u|^2 du dy \right|^p \right)^{\frac{1}{p}} \\ &\leq \|h'\|^2 \exp\left((2p-1)\|h'\|^2(t-\theta)\right)(t-\theta). \end{aligned} \quad (3.15)$$

Then (3.12) follows from (3.14) and (3.15) since

$$\begin{aligned} (E \|D\xi_t - D\xi_s\|_H^{2p})^{\frac{1}{p}} &= \left[E \left(\int_0^T |\mathbf{1}_{[s,t]}(\theta) + I_\theta^t(h' D_\theta \xi.)|^2 d\theta \right)^p \right]^{\frac{1}{p}} \\ &\leq \int_0^T \left(E |\mathbf{1}_{[s,t]}(\theta) + I_\theta^t(h' D_\theta \xi.)|^{2p} \right)^{\frac{1}{p}} d\theta \\ &\leq 2(t-s) + 2 \int_s^t \left(E |I_\theta^t(h' D_\theta \xi.)|^{2p} \right)^{\frac{1}{p}} d\theta \\ &\leq 2 \left(1 + \|h'\|^2 \exp\left((2p-1)\|h'\|^2(t-s)\right) \right) (t-s). \end{aligned}$$

For moments of $D^2(\xi_t - \xi_s)$, from (3.14) we have

$$D_{\eta,\theta}^2(\xi_t - \xi_s) = -D_\eta I_\theta^t(h' D_\theta \xi.) = I_\eta^t(h'' D_\theta \xi. D_\eta \xi.) - I_\eta^t(h' D_{\eta,\theta}^2 \xi.). \quad (3.16)$$

In a similar way as above we can get (3.13). ■

Next we derive some estimates for the density $p^W(r, x; t, y)$ of the conditional transition probability defined in (1.5). Denote

$$u_t \equiv \frac{D\xi_t}{\|D\xi_t\|_H^2}. \quad (3.17)$$

The next two lemmas give estimates of the divergence of u_t and $u_t - u_s$, which are important to derive the moment estimates of $p^W(r, x; t, y)$.

Lemma 3.3 *For any $p \geq 1$ and $0 \leq r < t \leq T$, we have*

$$\|\delta(u_t)\|_p \leq C(t-r)^{-\frac{1}{2}}. \quad (3.18)$$

Proof. Using the estimate (2.1) we obtain

$$\begin{aligned} \|\delta(u_t)\|_p &= (E|\delta(u_t)|^p)^{\frac{1}{p}} \leq [E(E^B|\delta(u_t)|^p)]^{\frac{1}{p}} \\ &\leq C_p(E[\|E^B u_t\|_H^p + (E^B\|Du_t\|_{H \otimes H}^p)])^{\frac{1}{p}} \\ &\leq C_p(\|u_t\|_H + \|Du_t\|_{H \otimes H}). \end{aligned}$$

We have

$$Du_t = \frac{D^2\xi_t}{\|D\xi_t\|_H^2} - 2 \frac{\langle D^2\xi_t, D\xi_t \otimes D\xi_t \rangle_{H \otimes H}}{\|D\xi_t\|_H^4},$$

and consequently $\|Du_t\|_{H \otimes H} \leq \frac{3\|D^2\xi_t\|_{H \otimes H}}{\|D\xi_t\|_H^2}$. Hence, for any positive number $\alpha, \beta > 1$ such that $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{1}{p}$, applying (3.7) and (3.8) we obtain (3.18):

$$\begin{aligned} \|\delta(u_t)\|_p &\leq C_p(\|D\xi_t\|_H^{-1} + 3\|D^2\xi_t\|_{L^\alpha(\Omega, H \otimes H)}\|D\xi_t\|_H^{-2}) \\ &\leq C(p, \|h'\|, \|h''\|, T)((t-r)^{-\frac{1}{2}} + (t-r)^{\frac{3}{2}}(t-r)^{-1}). \end{aligned}$$

This proves the lemma. ■

Lemma 3.4 *For $p \geq 1$, and $0 \leq r < s < t \leq T$,*

$$\|\delta(u_t - u_s)\|_{2p} \leq C(t-s)^{\frac{1}{2}}(s-r)^{-\frac{1}{2}}(t-r)^{-\frac{1}{2}}. \quad (3.19)$$

Proof. Using (3.14) we can write

$$u_t - u_s = \frac{D\xi_t}{\|D\xi_t\|_H^2} - \frac{D\xi_s}{\|D\xi_s\|_H^2} = A_1 + A_2 + A_3,$$

where

$$A_1 = D\xi_s \left(\frac{1}{\|D\xi_s\|_H^2} - \frac{1}{\|D\xi_t\|_H^2} \right), A_2 = \frac{\mathbf{1}_{[s,t]}(\theta)}{\|D\xi_t\|_H^2}, A_3 = \frac{I_\theta^t(h'D_\theta\xi.)}{\|D\xi_t\|_H^2}.$$

As a consequence, we have

$$\|\delta(u_t - u_s)\|_{2p} \leq \sum_{i=1}^3 \|\delta A_i\|_{2p}. \quad (3.20)$$

For simplicity we introduce the following notation

$$V_t \equiv \|D\xi_t\|_H, \quad N_t \equiv \|D^2\xi_t\|_{H \otimes H}, \quad Y_i = \|D^i(\xi_t - \xi_s)\|_{H^{\otimes i}}, \quad i = 1, 2.$$

Note that

$$\|A_1\|_H = \frac{\langle D\xi_t - D\xi_s, D\xi_t + D\xi_s \rangle}{\|D\xi_s\|_H \|D\xi_t\|_H^2} \leq Y_1 (V_t^{-2} + V_s^{-1}V_t^{-1}),$$

and

$$\begin{aligned} \|DA_1\|_{H \otimes H} &= \left\| D \left(\frac{D\xi_s \langle D\xi_t - D\xi_s, D\xi_t + D\xi_s \rangle}{\|D\xi_s\|_H^2 \|D\xi_t\|_H^2} \right) \right\|_{H \otimes H} \\ &\leq Y_1 N_s (V_s^{-2}V_t^{-1} + V_s^{-1}V_t^{-2}) + Y_2 (V_s^{-1}V_t^{-1} + V_t^{-2}) \\ &\quad + Y_1 (N_t + N_s) V_s^{-1}V_t^{-2} \\ &\quad + 2Y_1 [N_s (V_s^{-2}V_t^{-1} + V_s^{-1}V_t^{-2}) + N_t (V_t^{-3} + V_s^{-1}V_t^{-2})]. \end{aligned}$$

As a consequence, applying Lemma 2.1 and Hölder's inequality we get

$$\begin{aligned} \|\delta(A_1)\|_{2p} &\leq C \left(\|A_1\|_H + \|DA_1\|_{H \otimes H} \right) \\ &\leq C \|Y_1\|_{4p} \left(\|V_t^{-2}\|_{4p} + \|V_t^{-1}\|_{8p} \|V_s^{-1}\|_{8p} \right) \\ &\quad + C \|Y_1\|_{8p} \|N_s\|_{8p} \left(\|V_t^{-1}\|_{8p} \|V_s^{-2}\|_{8p} + \|V_s^{-1}\|_{8p} \|V_t^{-2}\|_{8p} \right) \\ &\quad + C \|Y_2\|_{4p} \left(\|V_t^{-1}\|_{8p} \|V_s^{-1}\|_{8p} + \|V_t^{-2}\|_{4p} \right) \\ &\quad + C \|Y_1\|_{8p} \left(\|N_s\|_{8p} + \|N_t\|_{8p} \right) \|V_t^{-2}\|_{8p} \|V_s^{-1}\|_{8p} \\ &\quad + 2C \|Y_1\|_{8p} \|N_s\|_{8p} \left(\|V_t^{-1}\|_{8p} \|V_s^{-2}\|_{8p} + \|V_t^{-2}\|_{8p} \|V_s^{-1}\|_{8p} \right) \\ &\quad + 2C \|Y_1\|_{8p} \|N_t\|_{8p} \left(\|V_t^{-3}\|_{4p} + \|V_t^{-2}\|_{8p} \|V_s^{-1}\|_{8p} \right). \end{aligned}$$

From Lemma 3.1 and Lemma 3.2 it follows that

$$\|\delta(A_1)\|_{2p} \leq C (t-s)^{\frac{1}{2}} (s-r)^{-\frac{1}{2}} (t-r)^{-\frac{1}{2}}. \quad (3.21)$$

Note that $\|A_2\|_H = \left\| \frac{\mathbf{1}_{[s,t]}(\theta)}{\|D\xi_t\|_H^2} \right\|_H = \|D\xi_t\|_H^{-2} (t-s)^{\frac{1}{2}}$ and

$$\|DA_2\|_{H \otimes H} \leq 2 \|D\xi_t\|_H^{-3} \|D^2\xi_t\|_{H \otimes H} (t-s)^{\frac{1}{2}}.$$

Then, by Lemma 2.1, Hölder's inequality and Lemma 3.1 we see that

$$\begin{aligned} \|\delta(A_2)\|_{2p} &\leq C \left(\|A_2\|_H + \|DA_2\|_{H \otimes H} \right) \\ &\leq C (t-s)^{\frac{1}{2}} \left(\|V_t^{-2}\|_{2p} + \|D^2\xi_t\|_{4p} \|V_t^{-1}\|_{4p} \right) \end{aligned}$$

$$\leq 2C (t-s)^{\frac{1}{2}} ((t-r)^{-1} + 1). \quad (3.22)$$

For the term A_3 , we apply Minkowski's inequality and the Burkholder-Davis-Gundy inequality and use (3.15). Thus for any $p \geq 1$,

$$\begin{aligned} & \left\| \left\| I_\theta^t (h' D_\theta \xi.) \right\|_H \right\|_{2p} = \left(E \left| \int_s^t I_\theta^t (h' D_\theta \xi.)^2 d\theta \right|^p \right)^{\frac{1}{2p}} \\ & \leq C_p \left(\int_s^t \left\| I_\theta^t (h' D_\theta \xi.) \right\|_{2p}^2 d\theta \right)^{\frac{1}{2}} \\ & \leq C_p \|h'\| \exp \left((2p-1) \|h'\|^2 (t-r) \right) (t-s)^{\frac{1}{2}}. \end{aligned} \quad (3.23)$$

From (3.16) it follows that

$$\begin{aligned} \|DA_3\|_{H \otimes H} & \leq \|D^2(\xi_t - \xi_s)\|_{H \otimes H} \|D\xi_t\|_H^{-2} \\ & \quad + 2 \|I_\theta^t (h' D_\theta \xi.)\|_H \|D^2\xi_t\|_{H \otimes H} \|D\xi_t\|_H^{-3}. \end{aligned}$$

Combining this with Lemma 2.1, Hölder's inequality, Lemma 3.2 and (3.23) we deduce

$$\begin{aligned} \|\delta(A_3)\|_{2p} & \leq C_p \left(\left\| \|A_3\|_H \right\|_{2p} + \|DA_3\|_{2p} \right) \\ & \leq C_p \left(\left\| \left\| I_s^t (h' D_\theta \xi.) \right\|_H \right\|_{2p} \|V_t^{-2}\|_{2p} + \|Y_2\|_{4p} \|V_t^{-2}\|_{4p} \right. \\ & \quad \left. + 2C_p \left\| \left\| I_s^t (h' D_\theta \xi.) \right\|_H \right\|_{4p} \|V_t^{-3}\|_{8p} \|N_t\|_{8p} \right) \\ & \leq C (t-s)^{\frac{1}{2}} (t-r)^{-1}. \end{aligned} \quad (3.24)$$

Substituting (3.21), (3.22) and (3.24) into (3.20) yields (3.19). ■

Now we provide the moment estimates for the conditional transition probability density $p^W(r, x; t, y)$.

Lemma 3.5 *Let $c = 1 \vee \|h\|^2$. For any $0 \leq r < t \leq T$, $y \in \mathbb{R}$ and $p \geq 1$,*

$$\left(E |p^W(r, x; t, y)|^{2p} \right)^{\frac{1}{2p}} \leq 2 \exp \left(-\frac{(x-y)^2}{64pc(t-r)} \right) \|\delta(u_t)\|_{4p}. \quad (3.25)$$

Proof. By Lemma 2.2 we can write

$$p^W(r, x; t, y) = E^B(\mathbf{1}_{\{\xi_t > y\}} \delta(u_t)) = E^B[\mathbf{1}_{\{B_r^t + I_r^t(h) > y-x\}} \delta(u_t)], \quad (3.26)$$

where B_r^t and $I_r^t(h)$ are defined in (3.2). Then, (2.2) implies

$$\begin{aligned} & \left(E |p^W(r, x; t, y)|^{2p} \right)^{\frac{1}{2p}} \\ & \leq \left(E \left[(P^B(|B_r^t + I_r^t(h)| > |y-x|))^p (E^B |\delta(u_t)|^2)^p \right] \right)^{\frac{1}{2p}} \\ & \leq \|\delta(u_t)\|_{4p} \left(E (P^B(|B_r^t + I_r^t(h)| > |y-x|))^{2p} \right)^{\frac{1}{4p}}. \end{aligned} \quad (3.27)$$

Applying Chebyshev and Jensen's inequalities, we have for $p \geq 1$,

$$\begin{aligned}
& E \left| P^B \left(|B_r^t + I_r^t(h)| > |y - x| \right) \right|^{2p} \\
& \leq \exp \left(\frac{-2p(x-y)^2}{32pc(t-r)} \right) E \left| E^B \exp \frac{(B_r^t + I_r^t(h))^2}{32pc(t-r)} \right|^{2p} \\
& \leq \exp \left(\frac{-(x-y)^2}{16c(t-r)} \right) E \exp \frac{(B_r^t + I_r^t(h))^2}{16c(t-r)}. \tag{3.28}
\end{aligned}$$

Using the fact that for $0 \leq \nu < 1/8$ and Gaussian random variables X, Y ,

$$E e^{\nu(X+Y)^2} \leq E e^{2\nu(X^2+Y^2)} \leq \left(E e^{4\nu X^2} \right)^{\frac{1}{2}} \left(E e^{4\nu Y^2} \right)^{\frac{1}{2}} = (1 - 8\nu)^{-\frac{1}{2}},$$

and noticing that B_r^t and $I_r^t(h)$ are Gaussian, we have

$$E \exp \frac{(B_r^t + I_r^t(h))^2}{16c(t-r)} \leq \left(1 - \frac{1}{2c} \right)^{-\frac{1}{2}} \leq \sqrt{2}. \tag{3.29}$$

Combining (3.27)–(3.29), we get (3.25). ■

4 Hölder continuity in spatial variable

In this section, we obtain the Hölder continuity of $X_t(y)$ with respect to y . More precisely, we show that for $t > 0$ fixed, $X_t(y)$ is almost surely Hölder continuous in y with any exponent in $(0, 1/2)$. This result was proved in [3]. Here we provide a different proof based on Malliavin calculus. We continue to use the notations B_r^t , $I_r^t(h)$ (defined by (3.2)) and u_t (defined by (3.17)).

Proposition 4.1 *Suppose that $h \in H^2_2(\mathbb{R})$ and $X_0 = \mu \in L^2(\mathbb{R})$ is bounded. Then, for any $t \in (0, T]$, $\alpha \in (0, 1)$ and $p > 1$, there exists a constant C depending only on $p, T, \|h\|_{2,2}$ and $\|\mu\|_{L^2(\mathbb{R})}$ such that*

$$E |X_t(y_2) - X_t(y_1)|^{2p} \leq C(1 + t^{-p}) (y_2 - y_1)^{\alpha p}. \tag{4.1}$$

Proof. We will use the convolution representation (1.6), where the two terms $X_{t,1}(y)$ and $X_{t,2}(y)$ will be estimated separately.

We start with $X_{t,2}(y)$. Suppose $y_1 < y_2 \in \mathbb{R}$. Note that $\mathbf{1}_{\{\xi_t > y_1\}} - \mathbf{1}_{\{\xi_t > y_2\}} = \mathbf{1}_{\{y_1 < \xi_t \leq y_2\}}$ and

$$E^B \mathbf{1}_{\{y_1 < \xi_t \leq y_2\}} = P^B \{y_1 < \xi_t \leq y_2\} = \int_{y_1}^{y_2} p^W(r, x; t, z) dz.$$

Therefore by (3.26) we have

$$\begin{aligned}
& \left| p^W(r, x; t, y_1) - p^W(r, x; t, y_2) \right|^2 = \left| E^B [\mathbf{1}_{\{y_1 < \xi_t < y_2\}} \delta(u_t)] \right|^2 \\
& \leq E^B |\delta(u_t)|^2 \int_{y_1}^{y_2} p^W(r, x; t, z) dz.
\end{aligned}$$

Hence,

$$\begin{aligned} & \left(E \left| p^W(r, x; t, y_1) - p^W(r, x; t, y_2) \right|^{2(2p-1)} \right)^{\frac{1}{2p-1}} \\ & \leq \| \delta(u_t) \|_{4(2p-1)}^2 \int_{y_1}^{y_2} \| p^W(r, x; t, z) \|_{2(2p-1)} dz. \end{aligned} \quad (4.2)$$

Lemma 3.3 and Lemma 3.5 yield

$$\begin{aligned} & \int_{\mathbb{R}} \left(E \left| p^W(r, x; t, y_1) - p^W(r, x; t, y_2) \right|^{2(2p-1)} \right)^{\frac{1}{2p-1}} dx \\ & \leq C \int_{\mathbb{R}} \| \delta(u_t) \|_{4(2p-1)}^3 \int_{y_1}^{y_2} \exp \left(\frac{-(z-x)^2}{32(2p-1)c(t-r)} \right) dz dx \\ & \leq C (t-r)^{-1} (y_2 - y_1). \end{aligned} \quad (4.3)$$

On the other hand, the left hand side of (4.3) can be estimated differently again by using Lemma 3.5:

$$\begin{aligned} & \int_{\mathbb{R}} \left(E \left| p^W(r, x; t, y_1) - p^W(r, x; t, y_2) \right|^{2(2p-1)} \right)^{\frac{1}{2p-1}} dx \\ & \leq 2 \int_{\mathbb{R}} \sum_{i=1,2} \left(E \left| p^W(r, x; t, y_i) \right|^{2(2p-1)} \right)^{\frac{1}{2p-1}} dx \\ & \leq C_p \int_{\mathbb{R}} \sum_{i=1,2} \| \delta(u_t) \|_{4(2p-1)}^2 \exp \left(\frac{-(y_i-x)^2}{64pc(t-r)} \right) dx \leq C (t-r)^{-\frac{1}{2}}. \end{aligned} \quad (4.4)$$

Then (4.3) and (4.4) yield that for any $\alpha, \beta > 0$ with $\alpha + \beta = 1$

$$\int_{\mathbb{R}} \left(E \left| p^W(r, x; t, y_1) - p^W(r, x; t, y_2) \right|^{2(2p-1)} \right)^{\frac{1}{2p-1}} dx \leq C (t-r)^{-\alpha-\frac{1}{2}\beta} (y_2 - y_1)^\alpha. \quad (4.5)$$

Since μ is bounded, it follows from [3, Lemma 4.1] that

$$\begin{aligned} & E \left| \int_0^t \int_{\mathbb{R}} (p^W(r, x; t, y_2) - p^W(r, x; s, y_1))^2 Z(dr dx) \right|^{2p} \\ & \leq C \left(E \left| \int_0^t \int_{\mathbb{R}} (p^W(r, x; t, y_2) - p^W(r, x; s, y_1))^2 dr dx \right|^{2p-1} \right)^{\frac{p}{2p-1}}, \end{aligned} \quad (4.6)$$

for any $p \geq 1$, $0 \leq s \leq t \leq T$ and $y_1, y_2 \in \mathbb{R}$. Then, applying Minkowski's inequality we obtain for any $0 < \alpha < 1$,

$$\begin{aligned} & (E |X_{t,2}(y_2) - X_{t,2}(y_1)|^{2p})^{\frac{1}{p}} \\ & \leq \int_0^t \int_{\mathbb{R}} \left(E \left| p^W(r, x; t, y_1) - p^W(r, x; t, y_2) \right|^{2(2p-1)} \right)^{\frac{1}{2p-1}} dx dr \\ & \leq C \int_0^t (t-r)^{-\alpha-\frac{1}{2}\beta} (y_2 - y_1)^\alpha dr \leq C (y_2 - y_1)^\alpha \end{aligned}$$

since $(t-r)^{-\alpha-\frac{1}{2}\beta} = (t-r)^{-(1+\alpha)/2}$ is integrable for all $0 < \alpha < 1$.

Now we consider $X_{t,1}(y)$ in (1.6). Applying Minkowski's inequality and using (4.2) with $2p-1$ replaced by p we get

$$\begin{aligned}
& E |X_{t,1}(y_2) - X_{t,1}(y_1)|^{2p} \\
& \leq \left(\int_{\mathbb{R}} \left(E |p(0, x; t, y_1) - p^W(0, x; t, y_2)|^{2p} \right)^{\frac{1}{2p}} \mu(x) dx \right)^{2p} \\
& \leq C \left\{ \int_{\mathbb{R}} \left(\int_{y_1}^{y_2} \|p^W(0, x; t, z)\|_{2p} dz \right)^{1/2} \|\delta(u_t)\|_{4p} \mu(x) dx \right\}^{2p} \\
& \leq C \|\delta(u_t)\|_{4p}^{2p} \|\mu\|_{L^2(\mathbb{R})}^{2p} \left(\int_{\mathbb{R}} \int_{y_1}^{y_2} \exp\left(-\frac{(z-x)^2}{64pct}\right) dz dx \right)^p \\
& \leq C \|\mu\|_{L^2(\mathbb{R})}^{2p} t^{-p} (y_2 - y_1)^p.
\end{aligned}$$

This completes the proof. ■

5 Hölder continuity in time variable

In this section we show that for any fixed $y \in \mathbb{R}$, $X_t(y)$ is Hölder continuous in t with any exponent in $(0, 1/4)$.

Proposition 5.1 *Suppose that $h \in H_2^2(\mathbb{R})$ and X_0 has a bounded density $\mu \in L^2(\mathbb{R})$. Then, for any $p \geq 1$, $0 \leq s < t \leq T$ and $y \in \mathbb{R}$,*

$$E |X_t(y) - X_s(y)|^{2p} \leq C(1 + t^{-p}) (t - s)^{\frac{p}{2} - \frac{1}{4}},$$

where the constant C depending only on $p, T, \|h\|_{2,2}$ and $\|\mu\|_{L^2(\mathbb{R})}$.

We need some preparations to prove the above result.

Suppose $0 < s < t$. We start by estimating $X_{s,2}(y)$ in (1.6) and we write

$$\begin{aligned}
X_{t,2}(y) - X_{s,2}(y) &= \int_0^s \int_{\mathbb{R}} (p^W(r, x; t, y) - p^W(r, x; s, y)) Z(dr dx) \\
&+ \int_s^t \int_{\mathbb{R}} p^W(r, x; t, y) Z(dr dx).
\end{aligned} \tag{5.1}$$

We are going to estimate the two terms separately.

Lemma 5.2 *For any $0 \leq s < t \leq T$, $y \in \mathbb{R}$ and $p \geq 1$, we have*

$$E \left(\int_0^s \int_{\mathbb{R}} (p^W(r, x; t, y) - p^W(r, x; s, y)) Z(dr dx) \right)^{2p} \leq C (t - s)^{\frac{p}{2} - \frac{1}{4}}. \tag{5.2}$$

Proof. From (3.26), we have for $0 < r < s < t \leq T$,

$$\begin{aligned}
p^W(r, x; t, y) - p^W(r, x; s, y) &= E^B [\mathbf{1}_{\{\xi_t > y\}} \delta(u_t) - \mathbf{1}_{\{\xi_s > y\}} \delta(u_s)] \\
&= E^B [(\mathbf{1}_{\{\xi_t > y\}} - \mathbf{1}_{\{\xi_s > y\}}) \delta(u_t) + \mathbf{1}_{\{\xi_s > y\}} \delta(u_t - u_s)],
\end{aligned}$$

Let $I_1 \equiv (\mathbf{1}_{\{\xi_t > y\}} - \mathbf{1}_{\{\xi_s > y\}}) \delta(u_t)$ and $I_2 \equiv \mathbf{1}_{\{\xi_s > y\}} \delta(u_t - u_s)$. Then (4.6) implies

$$\begin{aligned}
& E \left(\int_0^s \int_{\mathbb{R}} (p^W(r, x; t, y) - p^W(r, x; s, y)) Z(dr dx) \right)^{2p} \\
& \leq \left[E \left(\int_0^s \int_{\mathbb{R}} (E^B[I_1 + I_2])^2 dr dx \right)^{2p-1} \right]^{\frac{p}{2p-1}} \\
& \leq C \sum_{i=1,2} \left[E \left(\int_0^s \int_{\mathbb{R}} (E^B I_i)^2 dr dx \right)^{2p-1} \right]^{\frac{p}{2p-1}}.
\end{aligned} \tag{5.3}$$

First, we study the term I_1 . Note that

$$(\mathbf{1}_{\{\xi_t > y\}} - \mathbf{1}_{\{\xi_s > y\}})^2 = \mathbf{1}_{\{\xi_s \leq y < \xi_t\}} + \mathbf{1}_{\{\xi_t \leq y < \xi_s\}} =: A_1 + A_2.$$

Then we can write

$$\begin{aligned}
& \left[E \left(\int_0^s \int_{\mathbb{R}} E^B I_1^2 dr dx \right)^{2p-1} \right]^{\frac{1}{2p-1}} \\
& = \left[E \left(\int_0^s \int_{\mathbb{R}} E^B [(A_1 + A_2) \delta(u_t)]^2 dr dx \right)^{2p-1} \right]^{\frac{1}{2p-1}} \\
& \leq 2 \sum_{i=1,2} \left[E \left(\int_0^s \int_{\mathbb{R}} E^B [A_i \delta(u_t)]^2 dr dx \right)^{2p-1} \right]^{\frac{1}{2p-1}}.
\end{aligned} \tag{5.4}$$

Applying Minkowski, Jensen and Hölder's inequalities we deduce that for $i = 1, 2$ and for any conjugate pair (p_1, q_1)

$$\begin{aligned}
& \left[E \left(\int_0^s \int_{\mathbb{R}} E^B [A_i \delta(u_t)]^2 dr dx \right)^{2p-1} \right]^{\frac{1}{2p-1}} \\
& \leq \int_0^s \left(E \left| \int_{\mathbb{R}} A_i |\delta(u_t)|^2 dx \right|^{2p-1} \right)^{\frac{1}{2p-1}} dr \\
& \leq \int_0^s \left\| \left(\int_{\mathbb{R}} A_i dx \right)^{\frac{1}{p_1}} \left(\int_{\mathbb{R}} A_i |\delta(u_t)|^{2q_1} dx \right)^{\frac{1}{q_1}} \right\|_{2p-1} dr \\
& \leq \int_0^s \left\| \left| \int_{\mathbb{R}} A_i dx \right|^{\frac{1}{p_1}} \right\|_{2(2p-1)} \left\| \left(\int_{\mathbb{R}} A_i |\delta(u_t)|^{2q_1} dx \right)^{\frac{1}{q_1}} \right\|_{2(2p-1)} dr.
\end{aligned} \tag{5.5}$$

Notice that

$$\begin{aligned}
\{\xi_s \leq y < \xi_t\} &= \{y - B_r^t - I_r^t(h) < x \leq y - B_r^s - I_r^s(h)\}, \\
\{\xi_t \leq y < \xi_s\} &= \{y - B_r^s - I_r^s(h) < x \leq y - B_r^t - I_r^t(h)\}.
\end{aligned}$$

Then, for $i = 1, 2$, we have

$$\left| \int_{\mathbb{R}} A_i dx \right| = |B_s^t + I_s^t(h)|.$$

Hence for $p_1 = 1 - \frac{1}{2p}$,

$$\left\| \left| \int_{\mathbb{R}} A_i dx \right|^{\frac{1}{p_1}} \right\|_{2(2p-1)} \leq C (t-s)^{\frac{1}{2} - \frac{1}{4p}}. \quad (5.6)$$

On the other hand, we have

$$\begin{aligned} \{\xi_s \leq y < \xi_t\} &= \{B_r^s + I_r^s(h) \leq y - x < B_r^t + I_r^t(h)\} \\ &\subset \{|x - y| \leq |B_r^t + I_r^t(h)| + |B_r^s + I_r^s(h)|\}. \end{aligned}$$

Similarly

$$\{\xi_t \leq y < \xi_s\} \subset \{|x - y| \leq |B_r^t + I_r^t(h)| + |B_r^s + I_r^s(h)|\}.$$

Applying Chebyshev's inequality and (3.29), we deduce that for $i = 1, 2$,

$$\begin{aligned} E(A_i) &\leq EP^B \{|x - y| \leq |B_r^t + I_r^t(h)| + |B_r^s + I_r^s(h)|\} \\ &\leq \exp\left(\frac{-(x-y)^2}{32c(t-r)}\right) E \exp\left(\frac{|B_r^t + I_r^t(h)|^2}{16c(t-r)} + \frac{|B_r^s + I_r^s(h)|^2}{16c(s-r)}\right) \\ &\leq 2 \exp\left(-\frac{(x-y)^2}{32c(t-r)}\right). \end{aligned} \quad (5.7)$$

Using Minkowski and Hölder's inequalities, from (5.7) and Lemma 3.3 we obtain that for $q_1 = 2p \leq 2(2p-1)$,

$$\begin{aligned} &\left\| \left(\int_{\mathbb{R}} A_i |\delta(u_t)|^{2q_1} dx \right)^{\frac{1}{q_1}} \right\|_{2(2p-1)} \leq \left(\int_{\mathbb{R}} \|A_i |\delta(u_t)|^{2q_1}\|_{\frac{2(2p-1)}{q_1}} dx \right)^{\frac{1}{q_1}} \\ &\leq \left(\int_{\mathbb{R}} (EA_i)^{\frac{q_1}{4(2p-1)}} \|\delta(u_t)\|_{8(2p-1)}^{2q_1} dx \right)^{\frac{1}{q_1}} \leq C (t-r)^{\frac{1}{4p}-1}. \end{aligned} \quad (5.8)$$

Substituting (5.6) and (5.8) into (5.5) we obtain

$$\begin{aligned} &\left[E \left(\int_0^s \int_{\mathbb{R}} E^B [A_i \delta(u_t)]^2 dr dx \right)^{2p-1} \right]^{\frac{1}{2p-1}} \\ &\leq C (t-s)^{\frac{1}{2} - \frac{1}{4p}} \int_0^s (t-r)^{\frac{1}{4p}-1} dr \leq C (t-s)^{\frac{1}{2} - \frac{1}{4p}}. \end{aligned} \quad (5.9)$$

Combining (5.4) and (5.9), we have

$$\left[E \left(\int_0^s \int_{\mathbb{R}} E^B I_1^2 dr dx \right)^{2p-1} \right]^{\frac{1}{2p-1}} \leq C (t-s)^{\frac{1}{2} - \frac{1}{4p}}. \quad (5.10)$$

We turn into the term I_2 . From Lemma 2.3 we can deduce as in Lemma 3.5 that

$$\left(E (E^B I_2)^{2(2p-1)} \right)^{\frac{1}{2p-1}} \leq 2 \exp\left(\frac{-(x-y)^2}{32(2p-1)c(s-r)}\right) \|\delta(u_t - u_s)\|_{4(2p-1)}^2.$$

Then applying Minkowski's inequality and Lemma 3.4, we obtain

$$\begin{aligned}
& \left[E \left(\int_0^s \int_{\mathbb{R}} (E^B I_2)^2 dr dx \right)^{2p-1} \right]^{\frac{1}{2p-1}} \leq \int_0^s \int_{\mathbb{R}} \left(E (E^B I_2)^{2(2p-1)} \right)^{\frac{1}{2p-1}} dr dx \\
& \leq 2 \int_0^s \int_{\mathbb{R}} \exp \left(-\frac{(x-y)^2}{32(2p-1)(s-r)} \right) \|\delta(u_t - u_s)\|_{4(2p-1)}^2 dx dr \\
& \leq C(t-s) \int_0^s (s-r)^{\frac{1}{2}-1} (t-r)^{-1} dr \leq C(t-s)^{\frac{1}{2}-\frac{1}{4p}}, \tag{5.11}
\end{aligned}$$

where in the last step we used that $(t-r)^{-1} \leq (t-s)^{-\frac{1}{2}-\varepsilon} (s-r)^{-\frac{1}{2}+\varepsilon}$ for any $\varepsilon > 0$.

Substituting (5.10) and (5.11) in (5.3) we obtain (5.2). ■

Lemma 5.3 *For any $0 \leq s < t \leq T$ and any $y \in \mathbb{R}$ and $p \geq 1$, we have*

$$E \left(\int_s^t \int_{\mathbb{R}} p^W(r, x; t, y) Z(dr dx) \right)^{2p} \leq C(t-s)^{\frac{p}{2}}. \tag{5.12}$$

Proof. Since μ is bounded, it follows from [3, Lemma 4.1] that

$$E \left| \int_s^t \int_{\mathbb{R}} p^W(r, x; t, y)^2 Z(dr dx) \right|^{2p} \leq C \left(E \left| \int_s^t \int_{\mathbb{R}} p^W(r, x; t, y)^2 dr dx \right|^{2p-1} \right)^{\frac{p}{2p-1}}, \tag{5.13}$$

for any $p \geq 1$ and $y \in \mathbb{R}$. Applying Minkowski's inequality, Lemma 3.5 and Lemma 3.3 we obtain

$$\begin{aligned}
& \left[E \left(\int_0^s \int_{\mathbb{R}} |p^W(r, x; t, y)|^2 dr dx \right)^{2p-1} \right]^{\frac{1}{2p-1}} \\
& \leq C \int_s^t \int_{\mathbb{R}} \left(E |p^W(r, x; t, y)|^{2(2p-1)} \right)^{\frac{1}{2p-1}} dr dx \\
& \leq C \int_s^t \int_{\mathbb{R}} \exp \left(-\frac{(x-y)^2}{32c(t-r)} \right) \|\delta(u_t)\|_{4(2p-1)}^2 dr dx \\
& \leq C \int_s^t (t-r)^{\frac{1}{2}-1} dr \leq C(t-s)^{\frac{1}{2}}.
\end{aligned}$$

Then (5.12) follows immediately. ■

In summary of the above two lemmas, we get

Proposition 5.4 *For any $p \geq 1$, $0 \leq s < t \leq T$ and $y \in \mathbb{R}$, we have*

$$E |X_{t,2}(y) - X_{s,2}(y)|^{2p} \leq C(t-s)^{\frac{p}{2}-\frac{1}{4}}.$$

Now we consider $X_{t,1}(y)$. Note that

$$\begin{aligned}
& E |X_{t,1}(y) - X_{s,1}(y)|^{2p} = E \left| \int_{\mathbb{R}} (p^W(0, z; t, y) - p^W(0, z; s, y)) \mu(z) dz \right|^{2p} \\
& = E \left| \int_{\mathbb{R}} (E^B [\mathbf{1}_{\{\xi_t > y\}} \delta(u_t) - \mathbf{1}_{\{\xi_s > y\}} \delta(u_s)]) \mu(z) dz \right|^{2p}.
\end{aligned}$$

Then, similar to the proof for $X_{\cdot,2}(y)$ we get estimates for $X_{\cdot,1}(y)$.

Proposition 5.5 For any $p \geq 1$, $0 \leq s < t \leq T$ and any $y \in \mathbb{R}$, we have

$$E |X_{t,1}(y) - X_{s,1}(y)|^{2p} \leq C (1 + t^{-p}) (t - s)^{\frac{1}{2}p}. \quad (5.14)$$

Proof. Let $I_1 \equiv (\mathbf{1}_{\{\xi_t > y\}} - \mathbf{1}_{\{\xi_s > y\}}) \delta(u_t)$ and $I_2 \equiv \mathbf{1}_{\{\xi_s > y\}} \delta(u_t - u_s)$. Then,

$$E |X_{t,1}(y) - X_{s,1}(y)|^{2p} = E \left| \int_{\mathbb{R}} \mu(x) E^B [I_1 + I_2] dx \right|^{2p}.$$

Noticing that $|\mathbf{1}_{\{\xi_t > y\}} - \mathbf{1}_{\{\xi_s > y\}}| = \mathbf{1}_{\{\xi_s \leq y < \xi_t\}} + \mathbf{1}_{\{\xi_t \leq y < \xi_s\}} =: A_1 + A_2$, and applying Fubini's theorem, Jensen, Hölder and Minkowski's inequalities, we obtain

$$\begin{aligned} E \left| \int_{\mathbb{R}} \mu(x) E^B |I_1| dx \right|^{2p} &\leq \sum_{i=1,2} E \left| \int_{\mathbb{R}} \mu(x) E^B [A_i \delta(u_t)] dx \right|^{2p} \\ &\leq \sum_{i=1,2} E \left[\left(\int_{\mathbb{R}} |\mu(x) \delta(u_t)|^2 dx \right)^p \left| \int_{\mathbb{R}} A_i dx \right|^p \right] \\ &\leq \sum_{i=1,2} \left(\int_{\mathbb{R}} |\mu(x)|^2 \|\delta(u_t)\|_{4p}^2 dx \right)^p \left(E |B_s^t + I_s^t(h)|^{2p} \right)^{\frac{1}{2}} \\ &\leq C (1 + \|h\|^2) \|\mu\|_{L^2}^{2p} t^{-p} (t - s)^{\frac{1}{2}p}. \end{aligned}$$

For the term I_2 , using Minkowski's inequality, (3.25) and (3.19) with $r = 0$ we have

$$\begin{aligned} E \left| \int_{\mathbb{R}} \mu(x) E^B |I_2| dx \right|^{2p} &= \left(\int_{\mathbb{R}} |\mu(x)| \left(E |E^B \mathbf{1}_{\{\xi_s > y\}} \delta(u_t - u_s)|^{2p} \right)^{\frac{1}{2p}} dx \right)^{2p} \\ &\leq C \|\mu\|_{\infty}^{2p} \left(\int_{\mathbb{R}} \exp \left(-\frac{(x - y)^2}{32cs} \right) \|\delta(u_t - u_s)\|_{4p} dx \right)^{2p} \\ &\leq C \|\mu\|_{\infty}^{2p} t^{-p} (t - s)^p. \end{aligned}$$

Then we can conclude (5.14). ■

Proof of Proposition 5.1. It follows from Proposition 5.4 and Proposition 5.5. ■

Proof of Theorem 1.1. It follows from Proposition 4.1 and Proposition 5.1. ■

Acknowledgment The authors would like to thank Jie Xiong for bringing us of this topic.

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